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Analysis of Eigenvalues and Modal Interaction of Stochastic Systems

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An eigenvalue spectral analysis of stochastic engineering systems is presented. A comparative numerical study between approximations based on Monte Carlo sampling, a Taylor series-based perturbation approach, and the polynomial chaos representation is conducted. It is observed that the polynomial chaos representation gives more accurate estimates of the statistical moments than the perturbation method, especially for the higher modes. The differences of accuracy in the two methods are more pronounced as the system variability increases. Moreover, the chaos expansion gives a more detailed probabilistic description of the eigenvalues and the eigenvectors. In addition, a method for representing the statistical modal overlapping is presented that characterizes the statistical interaction between the various modes.

Nomenclature

$\boldsymbol{b}_{ik}$, $\boldsymbol{v}_{ik}$	=	n-dimensional vectors
c_{ik}, C_i^{kl}	=	deterministic scalar coefficients
e_i^l	=	random coefficient (scalar)
K	=	stiffness matrix
M	=	mass matrix
N_1, N_2	=	lower and upper index of deterministic modes
n	=	degrees of freedom of the system
P	=	maximum index of polynomial chaos
$\lambda_{\mathrm{av,mc}}$	=	mean eigenvalue computed by Monte Carlo simulation
$\lambda_{\mathrm{av,pert2}}$	=	mean eigenvalue computed from second-order perturbation
λ_{det}	=	eigenvalue of the deterministic system
λ_i	=	ith eigenvalue
$\boldsymbol{\xi}$, $\{\xi_i\}$	=	random vector with elements ξ_i
ξ_i	=	random variable
$\bar{\xi}_i$	=	mean value of the random variable ξ_i
σ_λ	=	standard deviation of the eigenvalue λ
$\boldsymbol{\phi}_i$	=	eigenvector
$\boldsymbol{\phi}_l^{(k)}$	=	kth chaos component of lth mode
$\bar{\boldsymbol{\phi}}_i$	=	ith mode of the deterministic system
ψ_i	=	ith polynomial chaos

Introduction

MOST physical systems exhibit some level of variability in their observed behavior. Such variability can be attributed to small changes in the manufacturing process or to contributions to the observed behavior from highly perturbative subscale phenomena. Attempts at capturing this variability in predictive models are faced with a number of challenges. In particular, additional sources of uncertainty quickly become significant in the prediction task. Insufficient details in the underlying physics, inadequate representation of this physics, as well as insufficient statistical representation of the natural variability, all contribute to the increased uncertainty in the predicted behavior of the system. Moreover, errors incurred in estimating the parameters of the physics models and in the numerical resolution of the prediction problem also exacerbate the uncertainty.

Probability theory provides a particular framework for characterizing the uncertainties in a manner that permits their quantification as well as that of their effects. Accordingly, the behavior of the system is described by a mathematical model with random parameters, such that the probabilistic character of the solution to these equations is representative, in some sense, of the observed scatter in the behavior of the actual system. Investigations into the properties of these probabilistic equations are essential for assessing their suitability at representing the physical problem.

Linear dynamic models constitute an important class of models used to describe the behavior of physical systems in a useful regime of operation. These models can be characterized in an efficient and accurate manner by reduced-order representations along their dominant eigenspaces. This useful algebraic property of these models is inherited by their probabilistic counterparts. In the probabilistic setting, however, model reduction on eigenspaces alters the probabilistic character of the predicted dynamics, thus further limiting the predictive accuracy of the associated model. Addressing this difficulty requires an understanding of the probabilistic properties of the eigenproblem associated with stochastic dynamic systems.

Analytical techniques are available for solving some particular classes of the random eigenvalue problems. For example, closed-form expressions of the probability density function (PDF) of the eigenvalues and the eigenvectors are available for Gaussian unitary ensemble and Gaussian orthogonal ensemble class of problems.[1] In general, however, and in particular for problems in structural dynamics, these analytical techniques are not applicable, and approximate solutions have to be sought. Once the statistics of the eigensolutions have been estimated, a constructive interpretation of their significance must be developed that will be meaningful in enhancing the performance of the system. Traditionally, the probabilistic content of the eigenvectors of a random system has been characterized by the mean and the covariance of its modal matrix.[2]

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This paper begins with a review and comparison of three approximations to the eigensolutions of stochastic systems. These are the perturbation method, Monte Carlo sampling, and approximations based on the polynomial chaos decomposition. Following that, a representation of the modal statistics is proposed that interprets the statistics of the modal vectors of the random system in terms of the modes of the mean or deterministic system. This offers a more detailed description and clearer prediction model of the behavior of the mode shapes of an uncertain system. This representation can also be used in efficient and accurate system reduction. Finally, a numerical example of a structural frame is used to demonstrate the concepts developed in the paper.

Random Eigenvalue Problem

Consider the n-dimensional algebraic random generalized symmetric eigenvalue problem

$$K(\boldsymbol{\xi})\phi = \lambda M(\boldsymbol{\xi})\phi, \qquad \lambda \in \mathbb{R}^{+}, \qquad \phi \in \mathbb{R}^{n}, \qquad K, M \in \mathbb{R}^{n\times n} \tag{1}$$

where $K(\boldsymbol{\xi})$ and $M(\boldsymbol{\xi})$ are $n \times n$ symmetric positive definite matrices functions of the random vector $\boldsymbol{\xi}$. The random variables in vector $\boldsymbol{\xi}$ denote the basic random variables in the problem that constitute the source of probabilistic uncertainty in the system. These represent, for example, elastic parameters that are modeled as random variables. In case such parameters are modeled as stochastic processes, the vector $\boldsymbol{\xi}$ represents a discretization of these processes with respect to some basis set, such as is associated, for instance, with the Karhunen–Loeve expansion (see Ref. 3). Clearly, the functional dependence of the matrices $K(\boldsymbol{\xi})$ and $M(\boldsymbol{\xi})$ on the random parameters $\boldsymbol{\xi}$ is deterministic, and for a specified probabilistic description of $\boldsymbol{\xi}$, the corresponding probabilistic descriptions of the random eigenvalues $\{\lambda_i\}$ and eigenvectors $\{\phi_i\}$ are sought. In general, the eigenvalues and eigenvectors of the system are fully characterized only by their joint probability distribution. In practice, estimates can be obtained of the first- and second-order joint statistics, as well as of the marginal PDF of the various quantities of interest.

Three methods are described next that provide varying degrees of approximation to these descriptions. These are the perturbation expansion, the polynomial chaos decomposition, and the Monte Carlo sampling.

In general, perturbation approaches attempt to quantify the variation in a dependent variable corresponding to a variation in an independent variable. A very general approach to express this correspondence between these two variations is achieved via a Taylor series development of the functional form, mapping the independent variable into the dependent one. The coefficients in that development are the partial derivatives of the functional form and, in the case of the eigenvalue problem, represent the derivatives of the eigenvalues and eigenvectors with respect to the uncertain parameters of the system. Whereas the first-order derivative can be interpreted as sensitivity coefficients around the nominal value at which the Taylor expansion is carried out, the higher-order derivatives require a probabilistic context for their interpretation. Although theoretically possible, it is generally quite impractical to develop perturbation-based estimates that require higher-order derivatives with respect to the random parameters.

In the Monte Carlo sampling (MC) technique, at first the basic random variables $\{\xi_i\}$ are generated. Each realization of the set of random variables $\{\xi_i\}$ is associated with a specific realization of the stochastic system, and the corresponding eigenproblem is solved for the set of $\{\lambda_i\}$ and $\{\phi_i\}$. Finally, the statistical moments and the PDF of the eigenvalues and eigenvectors are estimated from these realizations.

The polynomial chaos (PC) method describes a random vector or a process with respect to a set of Hermite polynomials (see Ref. 3). Note that this characterization of random vectors provides, implicitly, joint statistical information. The currently available technique of computing the PC expansion for random eigenvalue problem relies on the MC simulation technique.[4]

As expected, it is usually observed that deterministic or random perturbations of systems with closely spaced modes result in modal switching whereby the relative ordering changes for two modes associated with two distinct behaviors. In these cases, appropriate interpretation of the ordering of the perturbed modes is essential both for the proper application of associated statistical results as well as for the correct comparison of results from different approaches. The analyses presented in this paper all address the case of widely spaced modes, for which this issue is less significant. Clearly, the modal spacing is construed here to be relative to the strength of the perturbation.

Perturbation Approach

As already indicated, the perturbation approach relies on expressing the functional form of the eigenvalues and eigenvectors in terms of a small parameter, assumed here to be a perturbation in the value of the stiffness and mass parameters of the structure. Because these parameters are modeled as probabilistic objects, this procedure results in random perturbations, with the higher-order perturbation yielding information about the higher-order statistics of the solution. In a Taylor series development of the perturbation method, estimating higher-order statistics of the eigenvalues or eigenvectors, thus, requires the availability of estimates of the derivatives of the eigensolution with respect to the uncertain system parameters (see Refs. 5 and 6).

Because it is very complicated to calculate the higher-order derivatives, usually the series is truncated after the terms containing first- or second-order derivatives. The statistics of the eigensolution are evaluated in terms of the statistical moments of mass and stiffness properties.[2,7] Thus, representing the lth eigenvalue in a second-order expansion around the mean value of the q random parameters results in

$$\lambda_l = \lambda_l|_{\bar{\xi}} + \sum_{i=1}^{q} \left.\frac{\partial \lambda_l}{\partial \xi_i}\right|_{\bar{\xi}_i} (\xi_i - \bar{\xi}_i) + \frac{1}{2}\left[\left\{\sum_{i=1}^{q}(\xi_i - \bar{\xi}_i)\frac{\partial}{\partial \xi_i}\right\}^2 \lambda_l\right]_{\bar{\xi}} \tag{2}$$

which, on averaging, yields an expression for the mean value of the eigenvalue,

$$\bar{\lambda}_l = \lambda_l|_{\bar{\xi}} + \frac{1}{2}\sum_{i,j=1}^{q}\langle(\xi_i - \bar{\xi}_i)(\xi_j - \bar{\xi}_j)\rangle \left.\frac{\partial^2 \lambda_l}{\partial \xi_i \partial \xi_j}\right|_{\bar{\xi}_i,\bar{\xi}_j} \tag{3}$$

where $\langle . \rangle$ denotes the operation of mathematical expectation, and $\bar{\xi}_i$ and $\bar{\boldsymbol{\xi}}$ are the mean of the random variable and the vector, respectively. It is clear from this expression that correlation between the random variables $\{\xi_i\}$ introduces a discrepancy between the average of the eigenvalues and the eigenvalues of the average system. A first-order estimate of of the covariance between the rth and the sth eigenvalues can also be expressed as

$$\langle(\lambda_r - \bar{\lambda}_r)(\lambda_s - \bar{\lambda}_s)\rangle = \sum_{i,j=1}^{q} \left.\frac{\partial \lambda_r}{\partial \xi_i}\right|_{\bar{\xi}_i} \left.\frac{\partial \lambda_s}{\partial \xi_j}\right|_{\bar{\xi}_j} \langle \xi_i \xi_j \rangle \tag{4}$$

The covariance matrix among p eigenvalues involving q random parameters is then given as

$$[\mathrm{cov}(\lambda)] = \left[\frac{\partial \lambda}{\partial \xi}\right]_{\bar{\xi}} [\mathrm{cov}(\xi_i, \xi_j)] \left[\frac{\partial \lambda}{\partial \xi}\right]_{\bar{\xi}}^{T} \tag{5}$$

where the (r, s) element in $\mathrm{cov}(\lambda)$ is given by Eq. (4) and $\mathrm{cov}(\xi_i, \xi_j)$ refers to the covariance between random variables ξ_i and ξ_j. Moreover, in this case, the matrices $[\mathrm{cov}(\lambda)]$, $[\partial\lambda/\partial\xi]$, and $[\mathrm{cov}(\xi_i \xi_j)]$ are $(p \times p)$, $(p \times q)$, and $(q \times q)$, respectively.

As already seen, estimating the statistics of the eigensolution requires estimates of the derivatives of the eigenvalues with respect to the uncertain parameters of the system. When the sensitivities of the eigenvectors with respect to the uncertain parameters are neglected, and the chain rule for differentiation is applied, the sensitivity of the eigenvalues with respect to these parameter is given as[8,9]

$$\frac{\partial \lambda_i}{\partial \xi_k} = \phi_i^T \left[\frac{\partial K}{\partial \xi_k} - \lambda_i \frac{\partial M}{\partial \xi_k}\right]\phi_i \tag{6}$$

where it has been assumed that the eigenmodes are normalized with respect to the mass:

$$\phi_i^T M \phi_j = \delta_{ij} \tag{7}$$

Second-order derivatives of the eigenvalues can also be evaluated using a number of different methods.[10–13] The procedure presented here[14] follows a similar procedure for computing the first-order derivatives, resulting in

$$\frac{\partial^2 \lambda_i}{\partial \xi_j \partial \xi_k} = \phi_i^T \left[\frac{\partial^2 K}{\partial \xi_j \partial \xi_k} - \lambda_i \frac{\partial^2 M}{\partial \xi_j \partial \xi_k} - \frac{\partial \lambda_i}{\partial \xi_j} \frac{\partial M}{\partial \xi_k} - \frac{\partial \lambda_i}{\partial \xi_k} \frac{\partial M}{\partial \xi_j} \right] \phi_i + \phi_i^T \left[\frac{\partial}{\partial \xi_j} [K - \lambda_i M] \right] \frac{\partial \phi_i}{\partial \xi_k} + \phi_i^T \left[\frac{\partial}{\partial \xi_k} [K - \lambda_i M] \right] \frac{\partial \phi_i}{\partial \xi_j}$$

In this expression, the first-order sensitivity of the eigenvectors cannot be neglected and must be estimated. A number of algorithms have been developed to that end.[15] The particular method used here seems to be one of the computationally most efficient.[16] Accordingly, the eigenvector derivative is calculated by using the expression

$$\frac{\partial \phi_i}{\partial \xi_k} = \boldsymbol{v}_{ik} + c_{ik} \phi_i \tag{8}$$

where the vector $\boldsymbol{v}_{ik}$ is obtained as the solution to

$$D_i \boldsymbol{v}_{ik} = \boldsymbol{b}_{ik} \tag{9}$$

with

$$D_i = [K - \lambda_i M] \tag{10}$$

$$\boldsymbol{b}_{ik} = \frac{\partial \lambda_i}{\partial \xi_k} M \phi_i - \left[\frac{\partial K}{\partial \xi_k} - \lambda_i \frac{\partial M}{\partial \xi_k} \right] \phi_i \tag{11}$$

Given that D_i is a singular matrix, vector $\boldsymbol{v}_{ik}$ is normalized by setting its element with the same index as the greatest element in ϕ_i to zero in order. Finally, the scalar coefficients c_{ik} are calculated as

$$c_{ik} = -\phi_i^T M \boldsymbol{v}_{ik} - \frac{1}{2} \phi_i^T \frac{\partial M}{\partial \xi_k} \phi_i \tag{12}$$

The vector $\boldsymbol{v}_{ik}$ and the scalar c_{ik} computed from Eqs. (9–12) are substituted in Eq. (8) to compute the derivative of the eigenvector. The cases of repeated or closely spaced eigenvalues require a modified treatment and are not considered in the present paper.

PC Representation

The PC decomposition involves representing a second-order random variable or stochastic process with respect to a Hilbertian basis consisting of the multidimensional Hermite polynomials in orthonormal Gaussian variables.[3,17,18] Through a composition of transformations, it can, thus, be assumed, without loss of generality, that the basic random variables on which matrices K and M depend have already been transformed to a set of Gaussian random variables, denoted by $\boldsymbol{\xi}$. Generalizations of the PC decomposition to non-Gaussian kernels have recently been developed.[19,20] In the case of the random eigenproblem, and assuming that the eigenvalues and eigenvectors have a finite variance, they can then be expressed in their own PC decomposition[4] resulting in the following expressions for the lth eigenvector and eigenvalue:

$$\phi_l = \sum_{i=0}^{P} \psi_i \phi_l^{(i)}, \qquad \lambda_l = \sum_{i=0}^{P} \psi_i \lambda_l^{(i)}, \qquad \phi_l^{(i)} \in \mathbb{R}^n \tag{13}$$

where $\{\psi_i\}$ are the polynomial chaos basis that are zero-mean, multidimensional orthogonal polynomials in $\boldsymbol{\xi}$ such that

$$\psi_0 \equiv 1, \qquad \langle \psi_i \rangle = 0, \qquad i > 0, \qquad \langle \psi_i \psi_j \rangle = \delta_{ij} \langle \psi_i^2 \rangle \tag{14}$$

The coefficients of the expansion, $\phi_l^{(i)}$ and $\lambda_l^{(i)}$, are calculated by using their generalized Fourier coefficient expressions,

$$\phi_l^{(i)} = \langle \phi_l \psi_i \rangle \big/ \langle \psi_i^2 \rangle, \qquad \lambda_l^{(i)} = \langle \lambda_l \psi_i \rangle \big/ \langle \psi_i^2 \rangle \tag{15}$$

The denominator in the preceding expressions can be evaluated exactly,[3] whereas the numerator must be estimated, for example, by MC sampling. For each realization of $\{\xi_i\}$, in addition to solving the eigenvalue problem, the corresponding realization of ψ_i is also simultaneously computed as a polynomial form in the set $\{\xi_i\}$. The coefficients $\phi_l^{(i)}$ and $\lambda_l^{(i)}$ are then estimated by approximating the mathematical expectations using their finite sample statistical averages. Once the chaos coefficients have been evaluated, the statistical moments of the eigenvalues or eigenvectors can be computed directly, and realizations can be readily synthesized to estimate the corresponding PDFs. In particular, the first- and second-order statistical moments of the eigenvalues are obtained as

$$\bar{\lambda}_l \equiv \langle \lambda_l \rangle = \lambda_l^{(0)}, \qquad \langle (\lambda_l - \bar{\lambda}_l)^2 \rangle = \sum_{i=1}^{P} \langle \psi_i^2 \rangle \left(\lambda_l^{(i)} \right)^2 \tag{16}$$

Similarly, for the eigenvectors,

$$\bar{\phi}_l \equiv \langle \phi_l \rangle = \phi_l^{(0)} \tag{17}$$

A characterization of the higher-order statistics of modal vectors is discussed in the next section. Note that Eq. (13) permits the very efficient synthesis of statistical samples of the eigenvalues and eigenvectors once the polynomial chaos coefficients are known. Also note that the PC approach does not require explicit knowledge of the derivatives of the stiffness and mass matrices with respect to the random parameters.

Modal Statistics and Statistical Modal Interaction

As the variability in the system parameters increases, it can be expected that the probability of modal overlapping should become more significant. This is exacerbated for the system with closely spaced modes. Most analysis methods applicable to the random eigenproblem readily achieve a partial second-order statistical description of the random eigensolution.[2] This description is usually restricted to the marginal moments of the quantities of interest and is, therefore, not adequate for the analysis and characterization of modal interactions that involve joint behavior of eigensubspaces. A representation of the modal statistics is described next that provides a more complete statistical characterization of the modal overlapping.

In this methodology, the dynamic modes of the uncertain system are expressed in terms of the dynamic modes of the associated deterministic or mean system. This representation has a number of advantages. In particular, the concept of statistical modal overlapping, or statistical leakage, can be readily introduced. Because, for most systems, the eigenmodes of the deterministic system are used as the benchmark for design and decision making, describing the behavior of the stochastic system in terms of the modes of the deterministic system provides an easy interpretation of the system behavior in a probabilistic sense. This becomes more useful if the same system is analyzed for different types and levels of uncertainty.

Any statistical physical mode of an n-degree-of-freedom (DOF) system can be expressed as a linear combination of the deterministic physical modes as

$$\phi_l = \sum_{i=1}^{n} e_i^l \bar{\phi}_i, \qquad l = 1, \ldots, n \tag{18}$$

where the random variable e_i^l represents the random contribution of the ith deterministic mode to the lth mode of the stochastic system. It is assumed here that the normalization scheme used for the modal vectors is the same for both the deterministic and probabilistic eigenvectors. The lth mode of the uncertain system can be expanded in its polynomial chaos decomposition as

$$\phi_l = \sum_{k=0}^{P} \psi_k \phi_l^{(k)}, \qquad l = 1, \ldots, n \tag{19}$$

and the kth chaos component of the lth mode can be expressed as a linear combination of the deterministic physical modes as

$$\phi_l^{(k)} = \sum_{i=1}^{n} C_i^{kl} \bar{\phi}_i, \qquad l = 1, \ldots, n, \qquad k = 0, \ldots, P \tag{20}$$

where C_i^{kl} is the ith component of a vector obtained as the solution of an n-dimensional system of algebraic linear equations. Using Eqs. (19) and (20) yields

$$\phi_l = \sum_{k=0}^{P} \sum_{i=1}^{n} \psi_k C_i^{kl} \bar{\phi}_i = \sum_{i=1}^{n} \left(\sum_{k=0}^{P} \psi_k C_i^{kl} \right) \bar{\phi}_i, \qquad l = 1, \ldots, n \tag{21}$$

Comparing Eqs. (18) and (21) results in

$$e_i^l = \sum_{k=0}^{P} \psi_k C_i^{kl}, \qquad i, l = 1, \ldots, n \tag{22}$$

Clearly, the statistics of e_i^l provide a measure of the contribution of the deterministic modes to the modal properties of the variable system. In particular, the first two moments of e_i^l can be readily obtained as

$$\langle e_i^l \rangle = \sum_{k=0}^{P} \langle \psi_k \rangle C_i^{kl} = C_i^{0l} \tag{23}$$

$$\left\langle \left(e_i^l\right)^2 \right\rangle = \sum_{k=0}^{P} \left\langle \psi_k^2 \right\rangle \left(C_i^{kl}\right)^2 \tag{24}$$

Note that $\langle e_i^l \rangle$ is the projection of the 0th chaos component of the lth mode or the lth mean mode of the variable system found by chaos decomposition on the ith mode of the deterministic system. Clearly, this tends to a Kronecker delta δ_{il} as the magnitude of system variability decreases.

Clearly, the preceding analysis can be readily extended to the common situation where only a subspace of the whole spectrum is of interest.

Numerical Example

An example consisting of a 9-DOF frame is considered to demonstrate the foregoing analysis (Fig. 1). The frame consists of three rigid floors, each possessing two translational DOF in the direction of axes 1 and 2, as well as one in-plane rotational DOF. The frame is outfitted with energy dissipation devices that behave linearly once activated. The total restoring force in each floor is the sum of the restoring force from the columns and from the five restoring devices

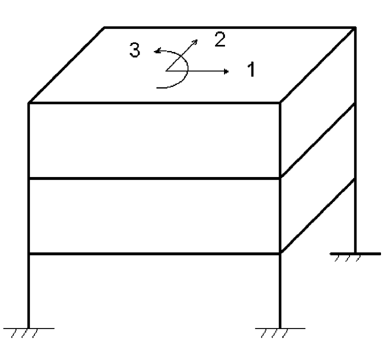

Fig. 1 Example 9-DOF frame considered for numerical study.

Table 1 Statistics of eigenvalues estimated by using various approximations[a]

			σ_λ			
λ_{det}	$\lambda_{\text{av,pert2}}$	$\lambda_{\text{av,mc}}$	Pert1	Pert2	PC2	MC
99.3	99.3	99.3	2.5	2.5	2.5	2.5
162.8	162.8	162.8	5.7	5.7	5.6	5.6
217.8	217.9	217.9	9.1	9.1	9.0	9.0
780.0	779.5	779.4	19.3	19.3	19.3	19.3
1277.7	1277.7	1277.7	44.6	44.6	44.3	44.3
1628.7	1627.6	1627.6	40.2	40.3	40.3	40.3
1710.1	1710.6	1710.7	71.1	71.1	70.7	70.7
2668.1	2668.1	2668.0	93.0	93.0	92.5	92.5
3570.9	3571.9	3572.0	148.4	148.4	147.9	147.9

[a]Standard deviation of the spring stiffnesses are $\sigma_1 = 5\%$ and $\sigma_2 = 5\%$.

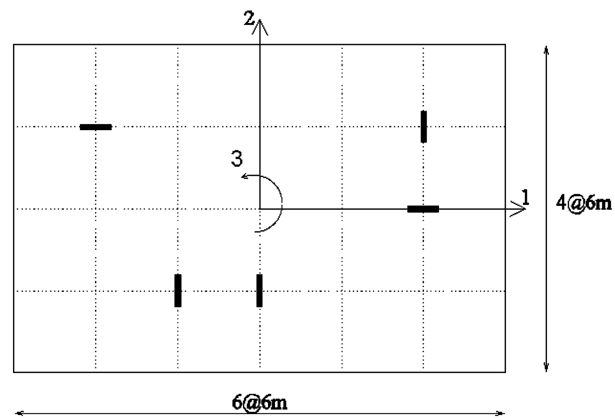

Fig. 2 Frame floor plan; location of restoring devices indicated by five bold bars.

placed in each floor at the spatial locations shown in Fig 2. The stiffnesses of the restoring devices in directions 1 and 2 are assumed to be independent random variables given, respectively, as

$$K_{r1} = \bar{K}_{r1} + \sigma_1 \xi_1, \qquad K_{r2} = \bar{K}_{r2} + \sigma_2 \xi_2 \tag{25}$$

where K_{r1} and K_{r2} are the stiffnesses in directions 1 and 2, respectively; $\bar{K}_{r1}$ and $\bar{K}_{r2}$ are the mean values; σ_1 and σ_2 are standard deviations; and ξ_1 and ξ_2 are two standard normal random variables. The nonzero contributions to the stiffness matrix from the frame columns are given by $K_{11} = 300$, $K_{14} = -150$, $K_{22} = 300$, $K_{25} = -150$, $K_{33} = 64{,}800$, $K_{36} - 32{,}400$, $K_{44} = 300$, $K_{47} = -150$, $K_{55} = 300$, $K_{58} = -150$, $K_{66} = 64{,}800$, $K_{69} = -32{,}400$, $K_{77} = 150$, $K_{88} = 150$, and $K_{99} = 32{,}400$, where K_{33}, K_{66}, K_{99} are in meganewtons meters and others are in meganewtons per meter. The diagonal mass matrix is given by $m_{11} = 1.0 \times 10^6$ kg, $m_{22} = 1.0 \times 10^6$ kg, $m_{33} = 156.0 \times 10^6\ \text{kg} \cdot \text{m}^2$, $m_{44} = 1.0 \times 10^6$ kg, $m_{55} = 1.0 \times 10^6$ kg, $m_{66} = 156.0 \times 10^6\ \text{kg} \cdot \text{m}^2$, $m_{77} = 1.0 \times 10^6$ kg, $m_{88} = 1.0 \times 10^6$ kg, and $m_{99} = 156.0 \times 10^6\ \text{kg} \cdot \text{m}^2$. Stiffness is also provided by the restoring devices each of which contributes an additional stiffness of 300 MN/m to each direction. The total contribution of the five devices in each floor is, therefore, given by $k_{11} = 600$ MN/m, $k_{22} = 900$ MN/m, $k_{33} = 64800\ \text{MN} \cdot \text{m}$, $k_{13} = -1800$ MN, and $k_{23} = 1800$ MN. The eigenvalues are tabulated in the first column of Tables 1 and 2. The wide spacings among the eigenvalues suggests the absence of modal switching and, consequently, permits as a first step a comparison of results from MC sampling, perturbation, and PC for well-separated modes.

In the present problem, where the basic random variables ξ_i are zero mean Gaussian and independent, considering up to the second-order terms in the perturbation technique, the mean eigenvalue is

$$\bar{\lambda}_l = \lambda_l|_{\bar{\xi}} + \frac{1}{2} \sum_{i=1}^{2} \left\langle \xi_i^2 \right\rangle \frac{\partial^2 \lambda_l}{\partial \xi_i^2} \bigg|_{\bar{\xi}_i} \tag{26}$$

Table 2 Eigenvalues of the statistics estimated by using various approximations[a]

			σ_λ				
λ_{det}	$\lambda_{\text{av,pert2}}$	$\lambda_{\text{av,mc}}$	Pert1	Pert2	PC2	PC4	MC
99.3	98.3	98.1	9.8	9.9	9.9	9.9	9.9
162.8	162.8	161.0	22.7	22.7	20.7	21.2	21.2
217.8	218.8	220.9	36.2	36.2	33.1	33.2	33.2
780.0	771.9	770.1	77.0	77.9	77.4	77.8	77.8
1277.7	1278.0	1263.7	178.2	178.2	162.9	166.0	166.4
1628.7	1611.8	1602.2	160.9	162.8	170.6	171.5	171.9
1710.1	1717.9	1739.8	284.2	284.4	250.5	251.6	251.7
2668.1	2668.7	2638.9	372.1	372.2	339.9	346.5	347.2
3570.9	3587.1	3620.9	593.5	594.0	541.9	544.6	544.9

[a]Standard deviation of spring stiffnesses are $\sigma_1 = 20\%$ and $\sigma_2 = 20\%$.

Table 3 Statistical leakage between the eigenvectors, as measured by $\langle (e_i^l)^2 \rangle$[a]

Deterministic system mode i	Variable system mode (l) 1	2	3	4	5	6	7	8	9
1	0.98	0.02	0.00	0.00	0.00	0.00	0.00	0.00	0.00
2	0.03	0.88	0.09	0.00	0.00	0.00	0.00	0.00	0.00
3	0.00	0.09	0.91	0.00	0.00	0.00	0.00	0.00	0.00
4	0.00	0.00	0.00	0.98	0.02	0.00	0.00	0.00	0.00
5	0.00	0.00	0.00	0.03	0.88	0.01	0.06	0.00	0.00
6	0.00	0.00	0.00	0.00	0.00	0.77	0.10	0.02	0.00
7	0.00	0.00	0.00	0.00	0.09	0.08	0.73	0.00	0.00
8	0.00	0.00	0.00	0.00	0.00	0.02	0.00	0.88	0.09
9	0.00	0.00	0.00	0.00	0.00	0.00	0.00	0.09	0.91

[a]Here $\sigma_1 = 20\%$ and $\sigma_2 = 20\%$.

and the variance becomes

$$\left\langle (\lambda_l - \bar{\lambda}_l)^2 \right\rangle = \sum_{i=1}^{2} \left(\frac{\partial \lambda_l}{\partial \xi_i}\bigg|_{\bar{\xi}_i} \right)^2 \left\langle \xi_i^2 \right\rangle + \frac{1}{4} \sum_{i=1}^{2} \left(\left\langle \xi_i^4 \right\rangle - \left\langle \xi_i^2 \right\rangle^2 \right) \times \left(\frac{\partial^2 \lambda_l}{\partial \xi_i^2}\bigg|_{\bar{\xi}_i} \right)^2 + \left\langle \xi_1^2 \xi_2^2 \right\rangle \left(\frac{\partial^2 \lambda_l}{\partial \xi_1 \partial \xi_2}\bigg|_{\bar{\xi}_1, \bar{\xi}_2} \right)^2 \tag{27}$$

where all derivatives are evaluated at the mean value of the random variables $\boldsymbol{\xi}$. For two combinations of standard deviations of the stiffening devices in directions 1 and 2, $\sigma_1 = \sigma_2 = 5\%$ and $\sigma_1 = \sigma_2 = 20\%$ of their mean values, the mean and standard deviation of the eigenvalues are computed using the perturbation method, the PC approach, and the MC simulation, respectively, and are presented in Tables 1 and 2. As already indicated, the first column in Tables 1 and 2, λ_{det}, is the eigenvalue of the deterministic system and also serves as the mean eigenvalue in the first-order perturbation approach. Here the deterministic system represents the system with mean stiffnesses. The second column in Tables 1 and 2, $\lambda_{\text{av,pert2}}$, is the mean eigenvalue computed from the second-order perturbation expansion, Eq. (26), and the third column, $\lambda_{\text{av,mc}}$ is the mean eigenvalue estimated from a MC sampling effort. This is also, clearly, the mean eigenvalue in chaos expansion. Here, σ_λ indicates the standard deviation of the eigenvalues. The columns labeled Pert1 and Pert2 report the σ_λ obtained from first- and second-order perturbations, respectively, PC2 and PC4 indicate second- and fourth-order chaos expansion. Finally, σ_λ computed from MC simulation are reported in the column labeled MC. For both the PC and the MC sampling methods, 50,000 samples for the system with lower variability (Table 1) and 100,000 samples for the system with higher variability (Table 2) were used. For computing the chaos coefficients, larger numbers of samples are required to estimate the higher-order coefficients compared to that required for estimating the lower-order ones.

A number of features are noted from Tables 1 and 2. From Table 1, where the spring stiffness variability is low, the results obtained from both the perturbation approach and the chaos expansion are very close with those obtained from MC sampling. As the system variability increases (Table 2), second-order perturbation expansion shows some improvement over the the first-order perturbation in estimating the statistics, especially the mean eigenvalues. However, overall, in this case, the results obtained from the PC approach are consistent with those obtained from MC simulation, whereas the perturbation scheme leads to significantly different results. This difference is more prominent for the higher modes. One potential source for this difference is the inherent limitation of the perturbation technique at capturing the large deviation of a function. For higher variability, the hypersurfaces describing the eigenvalues as a function of the parameters may intersect within the useful range of the parameter space, which results in modal phase switching for some realizations. In these cases, the difference of the eigenvalue ordering techniques in the perturbation and the sampling-based approaches contributes to the difference in the computed statistics. Note that, in the present example, increasing the order of the chaos expansion from second to fourth does not improve the result significantly.

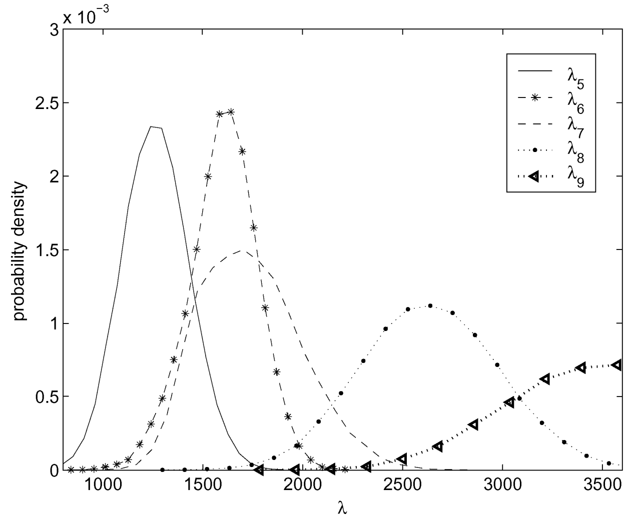

Fig. 3 PDF of eigenvalues (from fifth to ninth).

The modal overlapping is measured by the probabilistic description of the coefficients e_i^l. Table 3 shows values of $\langle (e_i^l)^2 \rangle$, with each column referring to the mode of the variable system, l, whereas each row corresponding to the mode of the deterministic system, i. The (i, l)th element in Table 3 refers to the contribution of the ith deterministic mode to the lth statistical dynamic mode, measured as $\langle (e_i^l)^2 \rangle$. Notice that the matrix is strongly diagonally dominant and is indeed a perturbation of the identity matrix. This is expected because in the deterministic limit, the eigenvectors are orthogonal. The width of the band varies with the closeness of the eigenvalues and level of uncertainty. In the present study, we have used the complete modal matrix of the deterministic system to evaluate the $\langle (e_i^l)^2 \rangle$, but the earlier mentioned banded form suggests even the truncated modal expansion could also be used here. The overlapping of sixth and seventh eigenvalues is shown in Fig. 3 and reinforces the preceding interpretation relating the statistics of e_i^l to the strength of modal interaction.

Conclusions

The problem of characterizing the eigenvalues and eigenvectors of random systems has been addressed. Three different methods, namely, perturbation, MC, and PC expansion are compared for efficiency, and a new insight is gained on the problem of modal interactions by further manipulations of the chaos representation. The system considered here has widely spaced modes at its mean. For such system, it is observed that, for small variability, all three methods yield similar results for the standard deviations of the eigenvalues. However, as the variability increases, the PC yield results that are closer to the MC simulations than the perturbation method, especially for higher modes. The difference originates from the truncation of the perturbation expression after the second-order term. This truncation is typically governed more by analytical convenience than by computational necessity, a difficulty not encountered with the PC representations.

A few features are worth noting with regard to the polynomial chaos approach as presented in this paper. The approach involves two steps. The first steps consists of postulating a representation of the eigenvalues and eigenvectors. In the present case, this step corresponds to choosing a coordinate system, or a basis in a vector space. The second step consists of computing the representation, or equivalently, the coordinates of the eigenvalues and eigenvectors with respect to the chosen coordinate system. Statistical sampling is used in the present paper for achieving this second step, with the resulting estimates of modal quantities being dependent on the size of the statistical sample.

It is noted, finally, that the significance of the probabilistic uncertainty on the statistical characterization of the modes increases both as the modal separation decreases and as the modal frequency increases. The investigation of the stability of modal subspaces, as opposed to individual modes, under random perturbation may be a more useful indicator for the analysis of system behavior.

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References

[1]Soize, C., and Ghanem, R., "Physical Systems with Random Uncertainties: Chaos Representations with Arbitrary Probability Measure," *SIAM Journal on Scientific Computing*, Vol. 26, No. 2, 2004, pp. 395–410.

[2]Xiu, D., Lucor, D., Su, C.-H., and Karniadakis, G. E., "Stochastic Modeling of Flow–Structure Interactions Using Generalized Polynomial Chaos," *Journal of Fluids Engineering*, Vol. 124, No. 1, 2002, pp. 51–59.

[3]Hart, G. C., and Collins, J. D., "The Treatment of Randomness in Finite Element Modelling," *SAE Shock and Vibrations Symposium*, 1970, pp. 2509–2519.

[4]Brandon, J., "Derivation and Significance of Second-Order Modal Design Sensitivities," *AIAA Journal*, Vol. 22, No. 5, 1984, pp. 723, 724.

[5]Brandon, J., "Second-Order Design Sensitivities to Assess the Applicability of Sensitivity Analysis," *AIAA Journal*, Vol. 29, No. 1, 1991, pp. 135–139.

[6]Lin, J., and Parker, R. G., "Sensitivity of Planetary Gear Natural Frequencies and Vibration Models to Model Parameters," *Journal of Sound and Vibration*, Vol. 228, No. 1, 1999, pp. 109–128.

[7]Red-Horse, J., and Ghanem, R., "Polynomial Chaos Representation of the Random Eigenvalue Problem," *Proceedings of the 40th Structures, Structural Dynamics, and Materials Conference*, AIAA, Reston, VA, 1999.

[8]Collins, J. D., and Thomson, W. T., "The Eigenvalue Problem for Structural Systems with Statistical Properties," *AIAA Journal*, Vol. 7, No. 4, 1969, pp. 642–648.

[9]Godoy, L., Taroco, E., and Feijoo, R., "Second-Order Sensitivity Analysis in Vibration and Buckling Problems," *International Journal for Numerical Methods in Engineering*, Vol. 37, 1994, pp. 3999–4014.

[10]Friswell, M., "Calculation of Second and Higher Order Eigenvector Derivatives," *Journal of Guidance, Control, and Dynamics*, Vol. 18, No. 4, 1995, pp. 919–921.

[11]Mehta, M. L., *Random Matrices*, 2nd ed., Academic Press, San Diego, CA, 1991.

[12]Wiener, N., "Homogeneous Chaos," *American Journal of Mathematics*, Vol. 60, No. 4, 1938, pp. 897–936.

[13]Ghanem, R., and Spanos, P. D., *Stochastic Finite Elements: A Spectral Approach*, rev. ed., Dover, Mineola, NY, 2003.

[14]Nelson, R. B., "Simplified Calculation of Eigenvector Derivatives," *AIAA Journal*, Vol. 14, No. 9, 1976, pp. 1201–1205.

[15]Cameron, R. H., and Martin, W. T., "The Orthogonal Development of Non-linear Functionals in Series of Fourier–Hermite Functionals," *Annals of Mathematics*, Vol. 48, No. 2, 1947, pp. 385–392.

[16]Fox, R. L., and Kapoor, M. P., "Rates of Changes of Eigenvalues and Eigenvectors," *AIAA Journal*, Vol. 6, No. 12, 1968, pp. 2426–2429.

[17]Sutter, T. R., Camarda, C. J., Walsh, J. L., and Adelmans, H. M., "Comparison of Several Methods for Calculating Vibration Mode Shape Derivatives," *AIAA Journal*, Vol. 26, No. 12, 1988, pp. 1506–1511.

[18]Mills-Curran, W. C., "Calculation of Eigenvector Derivatives for Structures with Repeated Eigenvalues," *AIAA Journal*, Vol. 26, No. 7, 1988, pp. 867–871.

[19]Mills-Curran, W. C., "Comment on 'Eigenvector Derivatives for Structures with Repeated Eigenvalues,'" *AIAA Journal*, Vol. 28, No. 10, 1990, p. 1846.

[20]Wittrick, W. H., "Rates of Change of Eigenvalues with Reference to Buckling and Vibration Problems," *Journal of the Royal Aeronautical Society*, Vol. 66, Sept. 1962, pp. 590, 591.

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